Capital Allocation Between The Risky And The Risk-Free Asset

Chapter 7

Investment Decisions

- *capital allocation decision* = choice of proportion to be invested in risk-free versus risky assets
- *asset allocation decision* = choice of type of assets to invest in (e.g., bonds, real estate, stocks, foreign assets etc.)
- *security selection decision* = choice of which particular security to invest in

Allocating Capital: Risky & Risk Free Assets

- examine risk/return tradeoff
- demonstrate how different degrees of risk aversion will affect allocations between risky and risk free assets
- consider the optimal risky portfolio as given and analyze the allocation decision between "the" risky portfolio (treated as *one* asset) and the risk-free asset (T-bills)
- rate of return:

$$r = \frac{P_1 - P_0 + D_1}{P_0}$$

The Risk-Free Asset

- technically, the risk-free asset is default-free and without inflation risk (a price-indexed default-free bond)
- in practice, Treasury bills come closest, because:
  - short term means little interest-rate or inflation risk
  - default risk is practically zero, since the government would no default

Notation

- r_f = rate of return on the risk-free asset
- r_p = rate of return on the risky portfolio
- r_C = rate of return on the *complete* portfolio (including both the risk-free asset and the risky portfolio)
- y = proportion of the investment budget to be placed in the risky portfolio
- σ_p = standard deviation of the return on the risky portfolio
- σ_C = standard deviation of the return on the complete portfolio

Characterization of the Complete Portfolio

- rate of return

$$r_C = yr_p + (1 - y)r_f$$

- expected rate of return

$$E(r_C) = y\,E(r_p) + (1 - y)\,E(r_f) = y\,E(r_p) + (1 - y)r_f$$
$$= r_f + y[E(r_p) - r_f]$$

- variance

$$\sigma_C^{\ 2} = y^2\sigma_p^{\ 2} + (1 - y)^2 \cdot 0 + 2y(1 - y)\,\mathrm{Cov}(r_p, r_f)$$
$$= y^2\sigma_p^{\ 2}$$

- standard deviation

$$\sigma_C = y\sigma_p$$

Available Complete Portfolios

- solve for y:

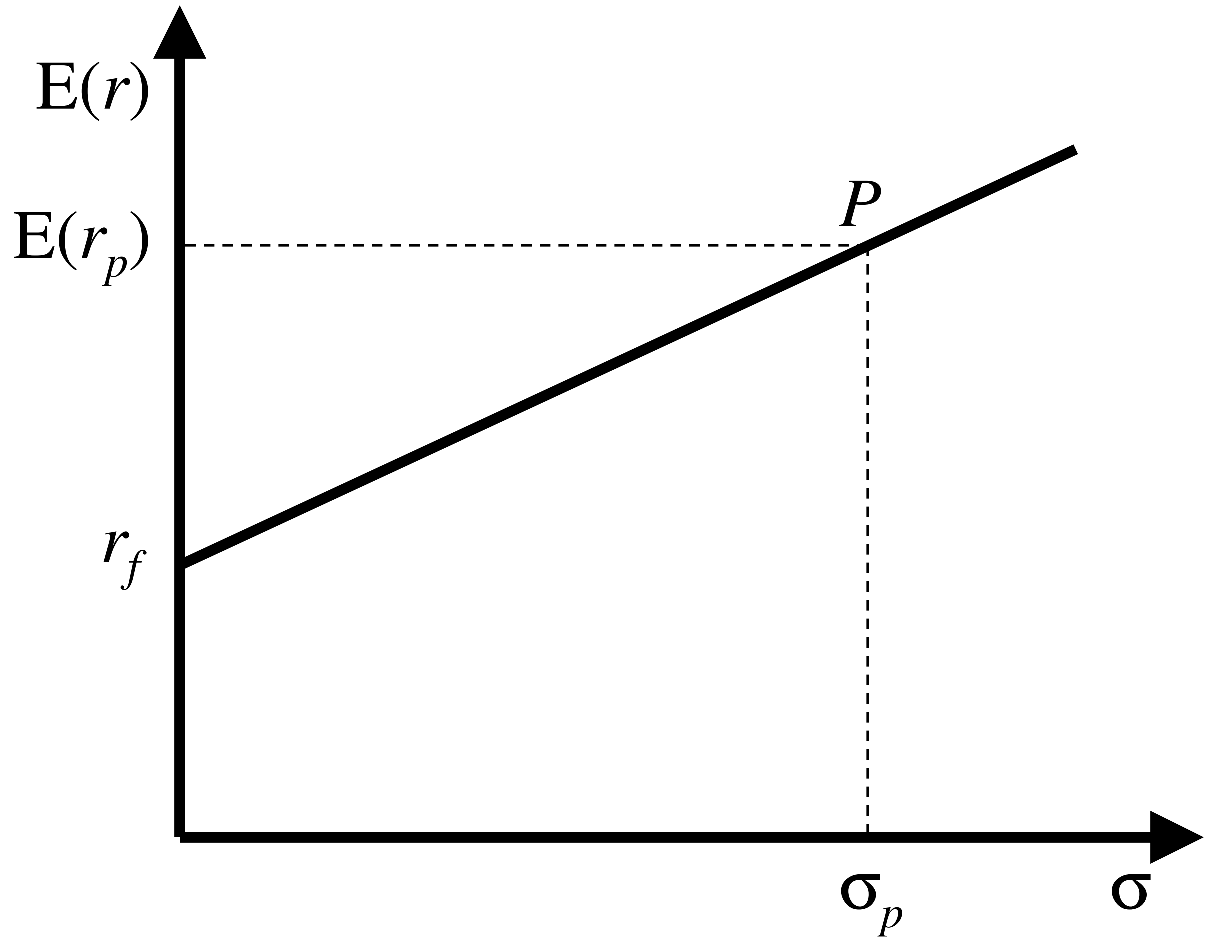
  $$y = \sigma_C / \sigma_p$$

- replace in the equation for the expected rate of return

  $$E(r_C) = r_f + \frac{\sigma_C}{\sigma_p}[E(r_p) - r_f] = r_f + \sigma_C \frac{[E(r_p) - r_f]}{\sigma_p}$$

- this defines a line in the mean-variance space – the capital allocation line (CAL)
- slope of CAL: $[\mathrm{E}(r_p) - r_f] / \sigma_p$

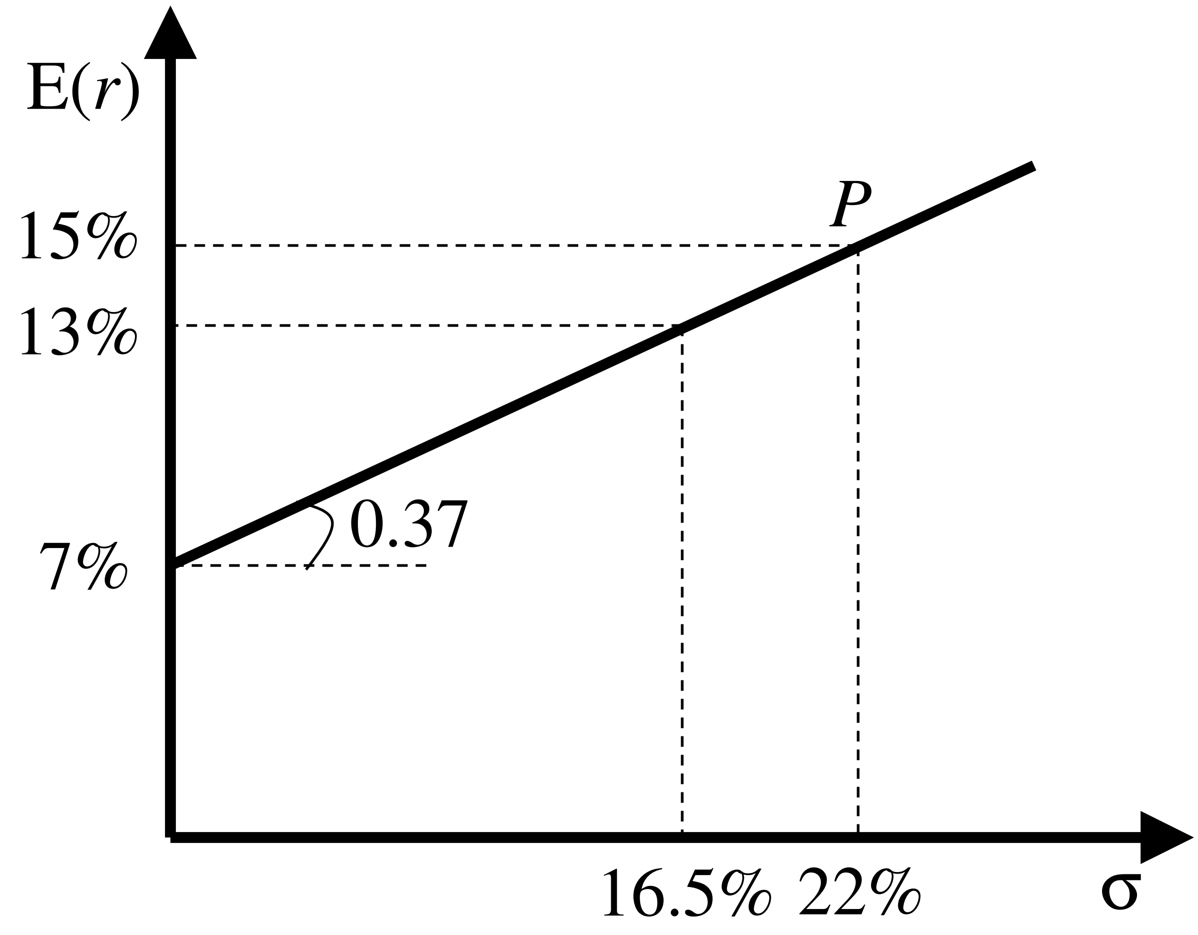
Capital Allocation Line



Example

- $r_f = 7\%$
- $E(r_p) = 15\%$
- $\sigma_p = 22\%$
- $y = 0.75$
- $E(r_C) = 0.75 \cdot 15\% + 0.25 \cdot 7\% = 13\%$
- $\sigma_C = y \cdot \sigma_p = 0.75 \cdot 22\% = 16.5\%$
- slope of CAL = $[E(r_p) - r_f] / \sigma_p = 8 / 22 = 0.37$

Capital Allocation Line – Example



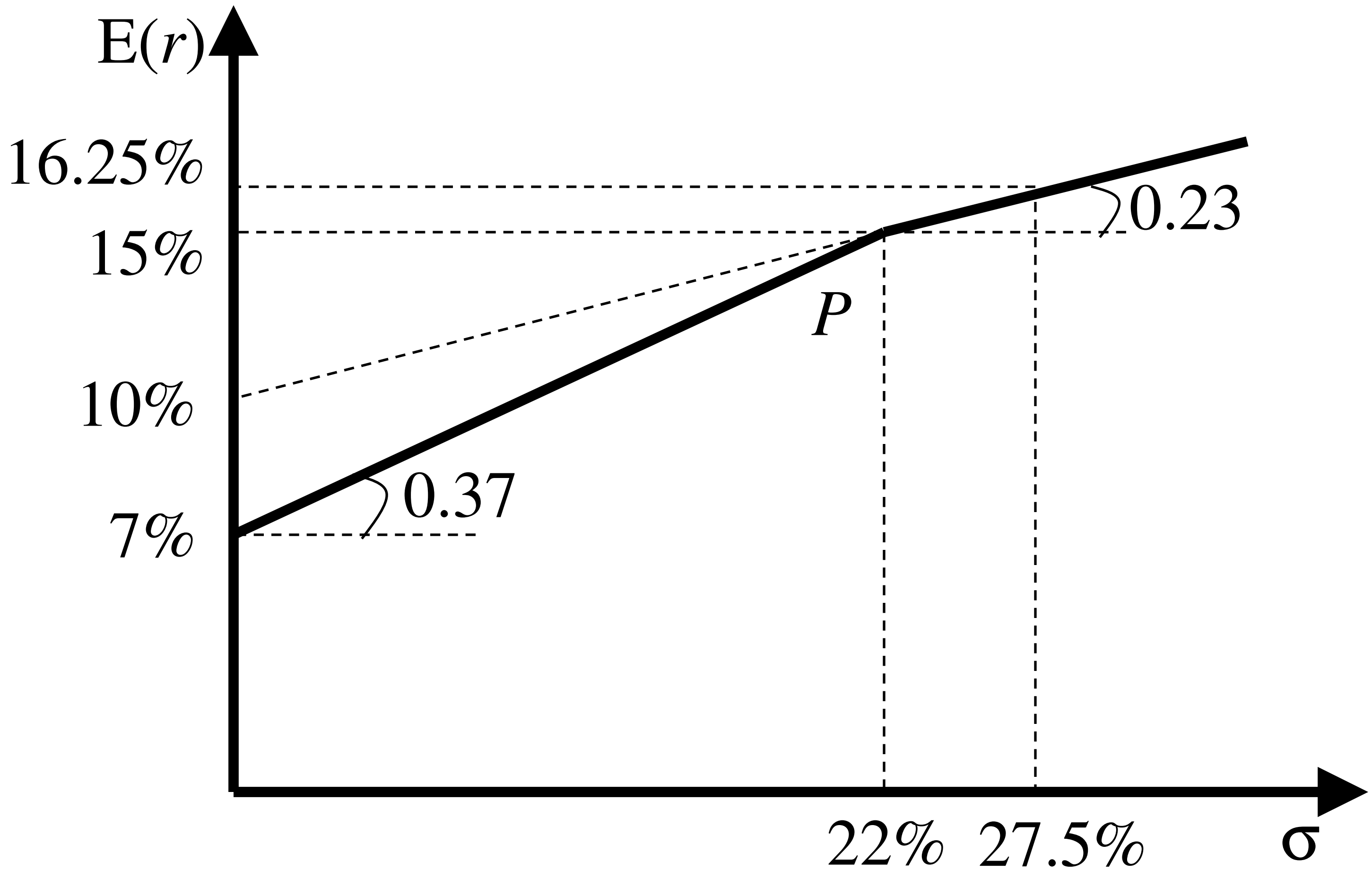
Capital Allocation Line with Leverage

- what happens if $y > 1$ (points to the right of P)?
- it means that there is *negative* investment in the risk-free asset ➔ the investor *borrowed* at the risk-free rate
- this is called *leveraged position in the risky asset* – some of the investment is financed by borrowing (e.g., buying on margin)
- the complete portfolio will have higher expected return, but also higher variance (risk)
- also, it is possible that the borrowing rate is higher than the lending rate (risk-free rate)

Example – Different Borrowing and Lending Rates

- $r_f = 7\%$
- $E(r_p) = 15\%$
- $\sigma_p = 22\%$
- $r_b = 10\%$
- $y = 1.25$
- $E(r_C) = 1.25 \cdot 15\% - 0.25 \cdot 10\% = 16.25\%$
- $\sigma_C = y \cdot \sigma_p = 1.25 \cdot 22\% = 27.5\%$
- slope of CAL (2) = $[E(r_p) - r_b] / \sigma_p = 5 / 22 = 0.23$

Capital Allocation Line – Example



Risk Aversion and Allocation

- higher levels of risk aversion lead to larger proportions of investment in the risk free asset (lower y)
- lower levels of risk aversion lead to larger proportions of investment in the portfolio of risky assets (higher y)
- willingness to accept high levels of risk for high levels of returns would result in leveraged combinations ($y > 1$)

Utility Function

- form of the utility function:

  $U = \mathrm{E}(r_C) - 0.005A\ \sigma_C^2$

- different values of A would cause different choices of the complete portfolio
- remember that
  - $\mathrm{E}(r_C) = r_f + y[\mathrm{E}(r_p) - r_f]$
  - $\sigma_C^2 = y^2\ \sigma_p^2$
- the utility function only as a function of y and known (expected) returns and variances:

  $U = r_f + y[\mathrm{E}(r_p) - r_f] - 0.005A\ y^2\ \sigma_p^2$

Optimal Complete Portfolio

- utility is maximized with respect to y:

  max $U = r_f + y[E(r_p) - r_f] - 0.005A\, y^2 \sigma_p^2$

- the solution is given by the first-order constraint (i.e., setting the derivative of U with respect to y equal to 0)

  $U' = [E(r_p) - r_f] - 0.005A \cdot 2y\, \sigma_p^2$

- solving for y gives the optimal choice of investment in the risky portfolio

$$y^* = \frac{E(r_p) - r_f}{0.01A\sigma_p^2}$$

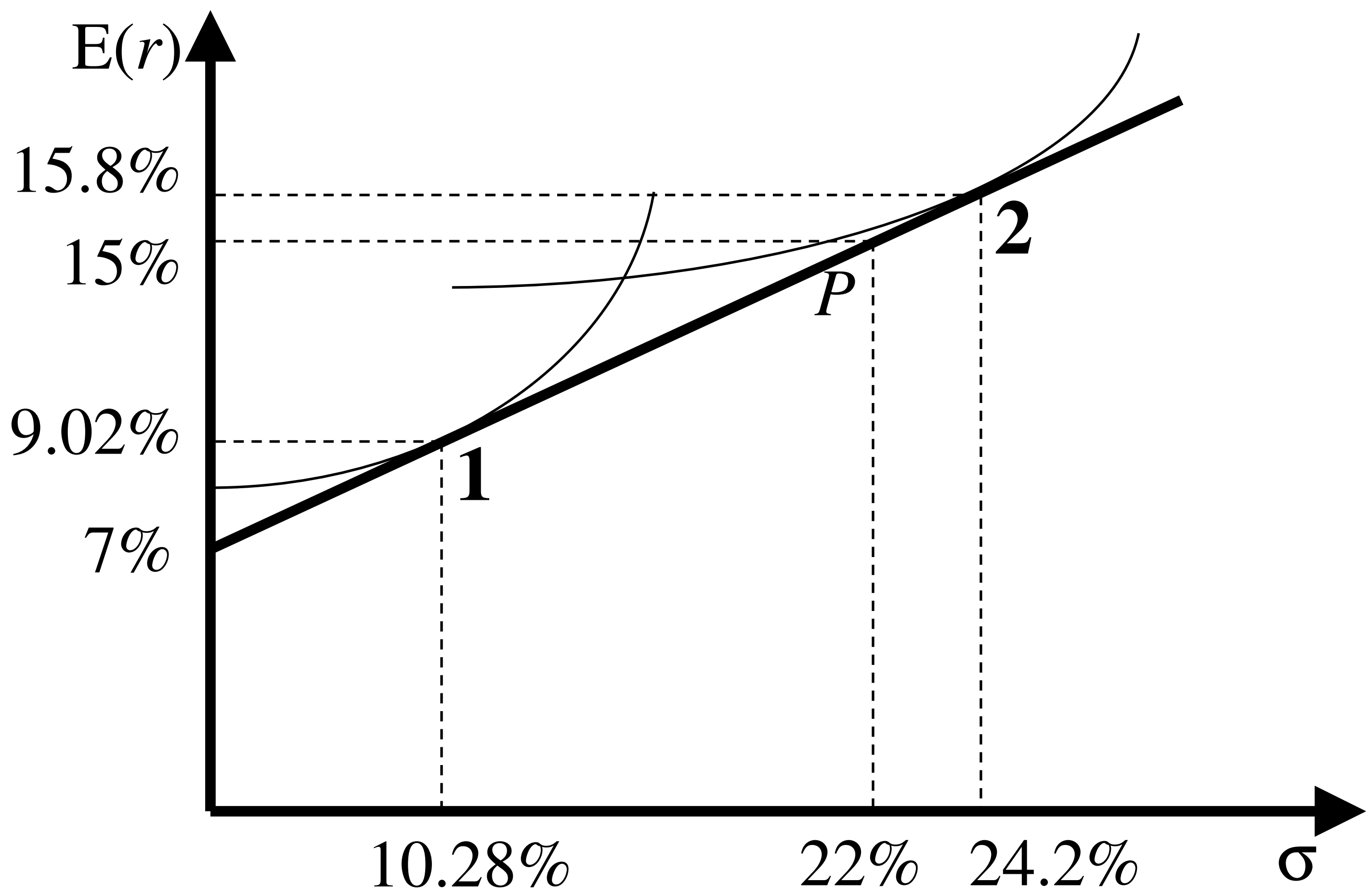
Optimal Complete Portfolio (cont.)

- optimal choice for an investor is the point of tangency of the highest indifference curve to the Capital Allocation Line ➔ slope of indifference curve is equal to the slope of the CAL
- borrowers (investors with $y > 1$) are less risk-averse than lenders (investors with $y \leq 1$)
- higher risk-aversion ➔ steeper indifference curve

Optimal Complete Portfolio–Example

- $r_f = 7\%$, $E(r_p) = 15\%$, $\sigma_p = 22\%$
- investor 1:
  - $A = 4$
  - $y = 8 / (0.01 \cdot 22^2 \cdot 4) = 0.41 = 41\%$
  - $E(r_C) = 0.41 \cdot 15\% + 0.59 \cdot 7\% = 10.28\%$
  - $\sigma_C = y \cdot \sigma_p = 0.41 \cdot 22\% = 9.02\%$
- investor 2:
  - $A = 1.5$
  - $y = 8 / (0.01 \cdot 22^2 \cdot 1.5) = 1.10 = 110\%$
  - $E(r_C) = 1.10 \cdot 15\% - 0.10 \cdot 7\% = 15.8\%$
  - $\sigma_C = y \cdot \sigma_p = 1.10 \cdot 22\% = 24.2\%$

Optimal Complete Portfolio and Risk Aversion



Capital Market Line

- we assumed that the investor chooses an optimal risky portfolio, which is given
- a *passive* strategy would be to invest in a broad portfolio, like a market index
- the resulting capital asset line is called *capital market line* (*CML*)