Optimal Risky Portfolios

Chapter 8
Diversification and Portfolio Risk

- *diversification* = investing in a larger number of assets

- sources of risk:
  - economy-wide factors (inflation, business cycles, exchange rates etc.)  ➔ *market (systematic) risk*
  - firm- or asset-specific factors  ➔ *idiosyncratic (nonsystematic) risk*

- if the firm-specific risk of the assets in the portfolio is independent, diversification can reduce the idiosyncratic risk (to zero), but *not* the market risk
Risk Reduction with Diversification

\[ \sigma \]

- Idiosyncratic risk

- Market risk

number of assets
Two-Security Portfolios

- **Objective:** analyze efficient diversification, i.e. obtaining the lower possible risk for any given level of expected return

- Consider two mutual funds, \( D \) (specialized in bonds and debt securities) and \( E \) (specialized in equity)

- The weight of mutual fund \( D \) in the portfolio is \( w_D \), and the weight of mutual fund \( E \) is \( w_E \), and their returns are \( r_D \) and \( r_E \)
Characterization of Two-Security Portfolios

- expected return of the portfolio:
  \[ E(r_p) = w_D E(r_D) + w_E E(r_E) \]

- variance of the portfolio:
  \[
  \sigma_p^2 = w_D^2 \sigma_D^2 + w_E^2 \sigma_E^2 + 2 w_D w_E \text{Cov}(r_D, r_E)
  \]
  \[
  = (w_D \sigma_D)^2 + (w_E \sigma_E)^2 + 2 (w_D \sigma_D) (w_E \sigma_E) \rho_{DE}
  \]
  \[
  = (w_D \sigma_D + w_E \sigma_E)^2 + 2 (w_D \sigma_D) (w_E \sigma_E) (\rho_{DE} - 1)
  \]
  \[
  = (w_D \sigma_D - w_E \sigma_E)^2 + 2 (w_D \sigma_D) (w_E \sigma_E) (\rho_{DE} + 1)
  \]

- if the two assets are not perfectly positively correlated, the standard deviation of the portfolio is less than the weighted average of the standard deviations of the assets
Two-Security Portfolios: Risk

- lowest variance achieved when assets perfectly negatively correlated
- in this case:

\[ \sigma_p^2 = (w_D \sigma_D - w_E \sigma_E)^2 \]

- can drive portfolio risk to zero with properly chosen weights:

\[ w_D = \frac{\sigma_E}{\sigma_D + \sigma_E} \]
\[ w_E = \frac{\sigma_D}{\sigma_D + \sigma_E} = 1 - w_D \]
Two-Security Portfolios: Risk (cont.)

- zero-variance is possible with positively correlated assets as well, but it would involve short-selling (e.g., \( w_D > 1 \) and \( w_E < 0 \))
- can trace out the portfolio opportunity set, the set of portfolios as the weights invested in the assets vary
- *minimum-variance portfolio* = portfolio in the opportunity set with minimum variance
- can also analyze how the shape of the portfolio opportunity set changes due to different correlations between the assets
Portfolio Opportunity Set

\[ \rho_{DE} = 1 \]

\[ -1 < \rho_{DE} < 1 \]

\[ \rho_{DE} = -1 \]
Three-Security Portfolios

- add a risk-free asset
- the CAL depends on the portfolio chosen as the risky portfolio
- remember the slope of the CAL (Sharpe ratio)

\[ S_A = \frac{E(r_A) - r_f}{\sigma_A} \]
Portfolio Opportunity Set with a Risk-free Asset
Optimal Risky Portfolio

- optimal choices of complete portfolios lie on the highest possible CAL
- hence, optimal risky portfolio is the tangency point of the highest CAL to the portfolio opportunity set
- the highest CAL also has the highest slope
- thus, the investor maximizes the slope:

\[
\max_{w_i} S_p = \frac{E(r_p) - r_f}{\sigma_p}
\]

- optimal complete portfolio given by investor indifference curve
Optimal Complete Portfolio
Markovitz Portfolio Selection Model

- many risky assets and a risk-free asset
- *minimum-variance frontier* = the set of portfolios with the lowest variance given an expected rate of return
- *efficient frontier* = the set of portfolios on the minimum-variance frontier, with expected return higher than that of the global minimum variance portfolio
- when short-sales are allowed, all individual assets lie *inside* the minimum-variance frontier
Minimum-Variance Frontier

Efficient Frontier

Global Minimum-Variance Portfolio

Individual Assets

Minimum-Variance Frontier

E(r)

σ
Portfolio Selection without a Risk-free Asset

\[
E(r) \quad \sigma
\]

\[
r_f
\]
Capital Allocation with Many Risky Assets

- add a risk-free asset
- again, the CAL depends on the portfolio chosen as *the* risky portfolio
- investors will choose the highest CAL, i.e., the CAL tangent to the efficient frontier
- again, this portfolio is the solution to the optimization problem of maximizing the slope of the CAL
Capital Allocation Line with Many Risky Assets

Efficient Frontier

\( E(r) \) vs. \( \sigma \)

\( r_f \)
Separation Property

- note that all investors will choose the same portfolio of risky assets (say $P$) for their capital allocation problem.
- this leads to the separation property = the portfolio choice problem can be broken down into two tasks:
  - choosing $P$, a technical matter (can be done by the broker)
  - deciding on the proportion to be invested in $P$ and in the risky asset (decided by the client)
Portfolio Selection with a Risk-free Asset
Portfolio Selection with Borrowing Constraints

![Graph showing portfolio selection with borrowing constraints]
Investment Opportunity Set with Different Borrowing Rate

\[ E(r) \]

\[ \sigma \]

\[ r_b \]

\[ r_f \]

\[ P_1 \]

\[ P_2 \]
Investment Opportunity Set with Different Borrowing Rate (cont.)

$E(r)$

$r_b$

$r_f$

$P_1$

$P_2$

$\sigma$
Portfolio Selection with Different Borrowing Rate

\[ E(r) \]

\[ \sigma \]

\[ r_b \]

\[ r_f \]