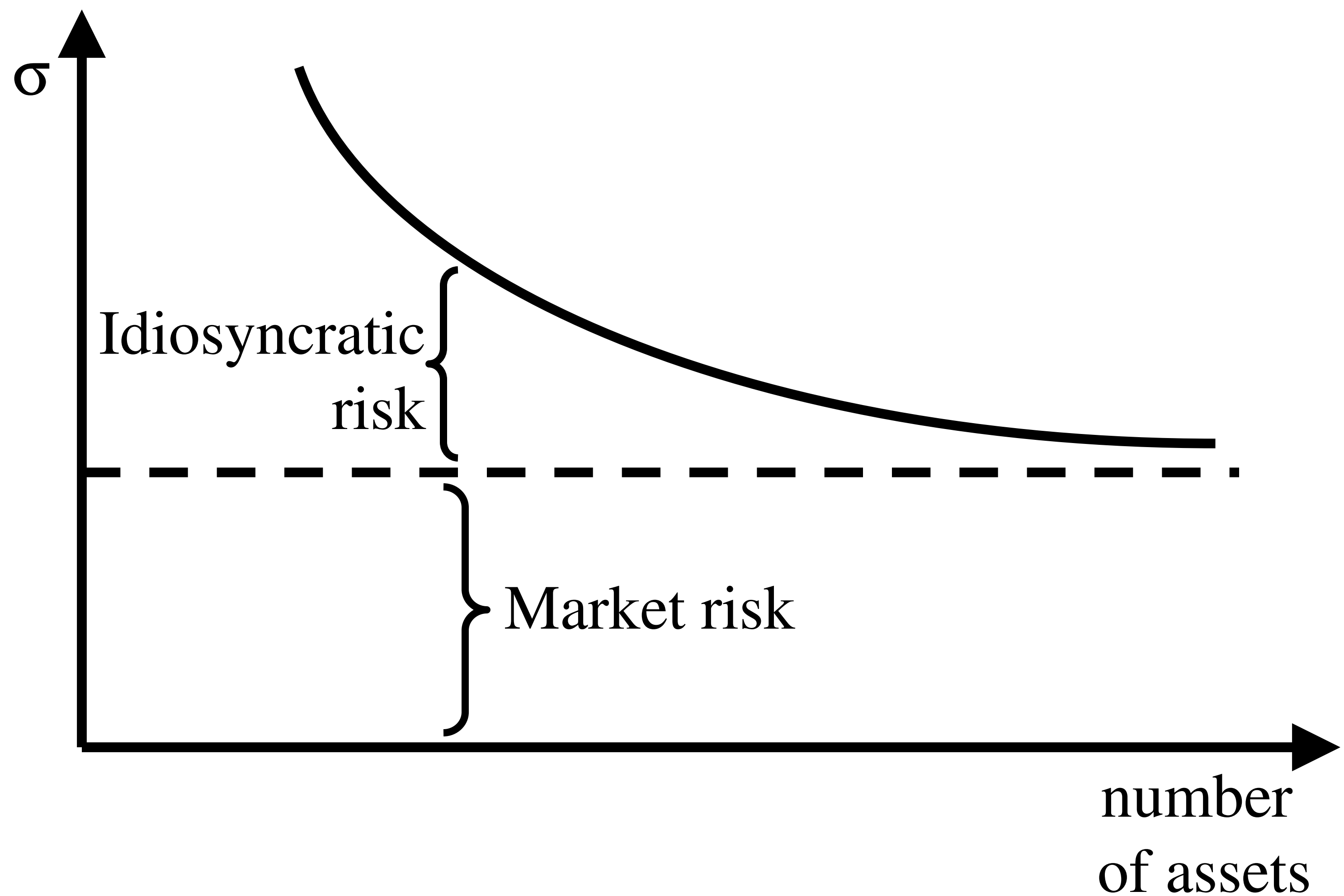
Optimal Risky Portfolios

Chapter 8

Diversification and Portfolio Risk

- *diversification* = investing in a larger number of assets
- sources of risk:
  - economy-wide factors (inflation, business cycles, exchange rates etc.) ➔ *market (systematic) risk*
  - firm- or asset-specific factors ➔ *idiosyncratic (nonsystematic) risk*
- if the firm-specific risk of the assets in the portfolio is independent, diversification can reduce the idiosyncratic risk (to zero), but *not* the market risk

Risk Reduction with Diversification



Two-Security Portfolios

- objective: analyze *efficient* diversification, i.e. obtaining the lower possible risk for any given level of expected return
- consider two mutual funds, *D* (specialized in bonds and debt securities) and *E* (specialized in equity)
- the weight of mutual fund *D* in the portfolio is w_D, and the weight of mutual fund *E* is w_E, and their returns are r_D and r_E

Characterization of Two-Security Portfolios

- expected return of the portfolio:

  $$E(r_p) = w_D\, E(r_D) + w_E\, E(r_E)$$

- variance of the portfolio:

  $$\begin{aligned}\sigma_p^2 &= w_D^2\, \sigma_D^2 + w_E^2\, \sigma_E^2 + 2\, w_D\, w_E\, \mathrm{Cov}(r_D, r_E) \\ &= (w_D\sigma_D)^2 + (w_E\sigma_E)^2 + 2\,(w_D\sigma_D)\,(w_E\sigma_E)\,\rho_{DE} \\ &= (w_D\sigma_D + w_E\sigma_E)^2 + 2\,(w_D\sigma_D)\,(w_E\sigma_E)\,(\rho_{DE} - 1) \\ &= (w_D\sigma_D - w_E\sigma_E)^2 + 2\,(w_D\sigma_D)\,(w_E\sigma_E)\,(\rho_{DE} + 1)\end{aligned}$$

- if the two assets are not perfectly positively correlated, the standard deviation of the portfolio is less than the weighted average of the standard deviations of the assets

Two-Security Portfolios: Risk

- lowest variance achieved when assets perfectly negatively correlated
- in this case:

$$\sigma_p^2 = (w_D\sigma_D - w_E\sigma_E)^2$$

- can drive portfolio risk to zero with properly chosen weights:

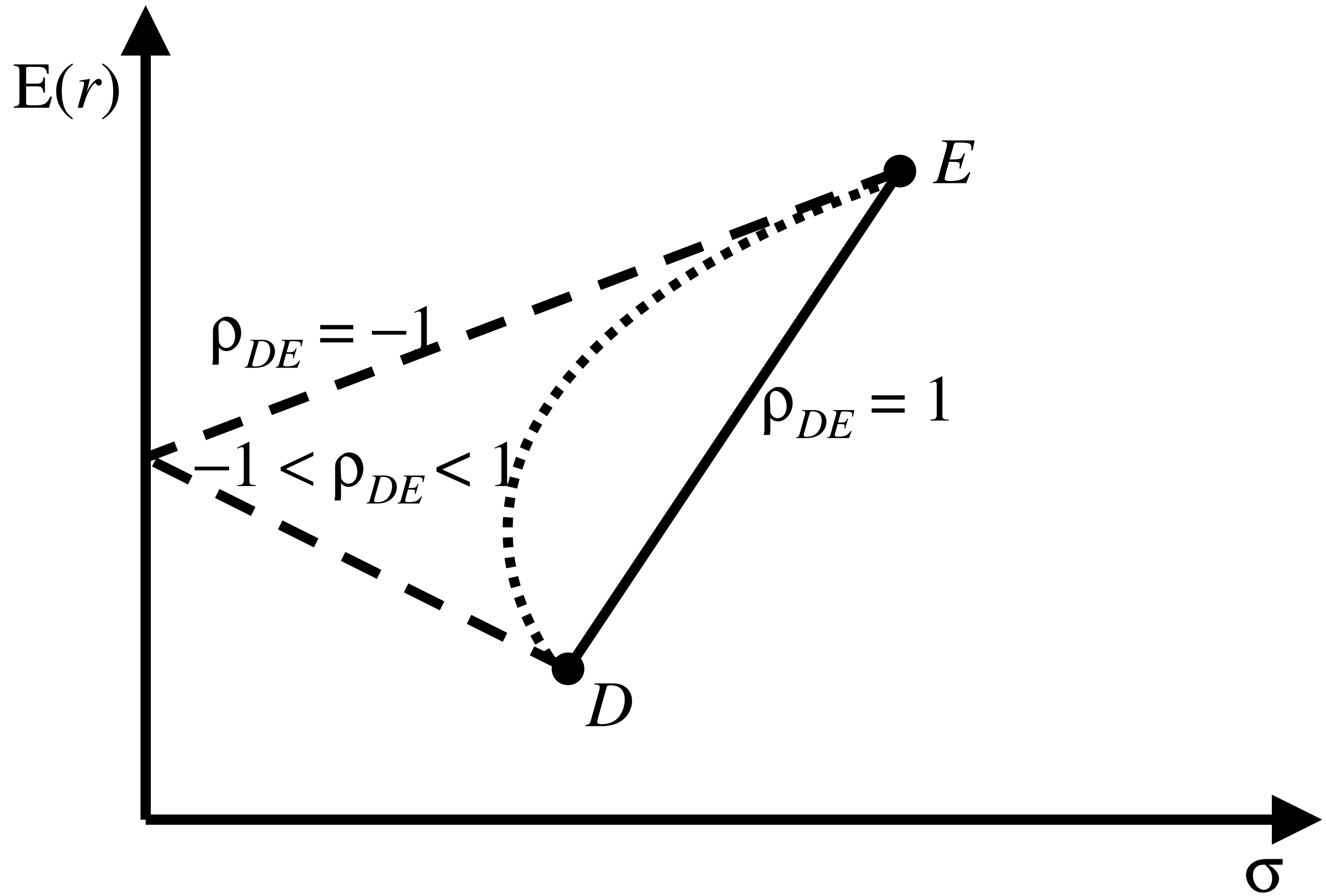
$$w_D = \frac{\sigma_E}{\sigma_D + \sigma_E}$$

$$w_E = \frac{\sigma_D}{\sigma_D + \sigma_E} = 1 - w_D$$

Two-Security Portfolios: Risk (cont.)

- zero-variance is possible with positively correlated assets as well, but it would involve short-selling (e.g., $w_D > 1$ and $w_E < 0$)
- can trace out the *portfolio opportunity set*, the set of portfolios as the weights invested in the assets vary
- *minimum-variance portfolio* = portfolio in the opportunity set with minimum variance
- can also analyze how the shape of the portfolio opportunity set changes due to different correlations between the assets

Portfolio Opportunity Set

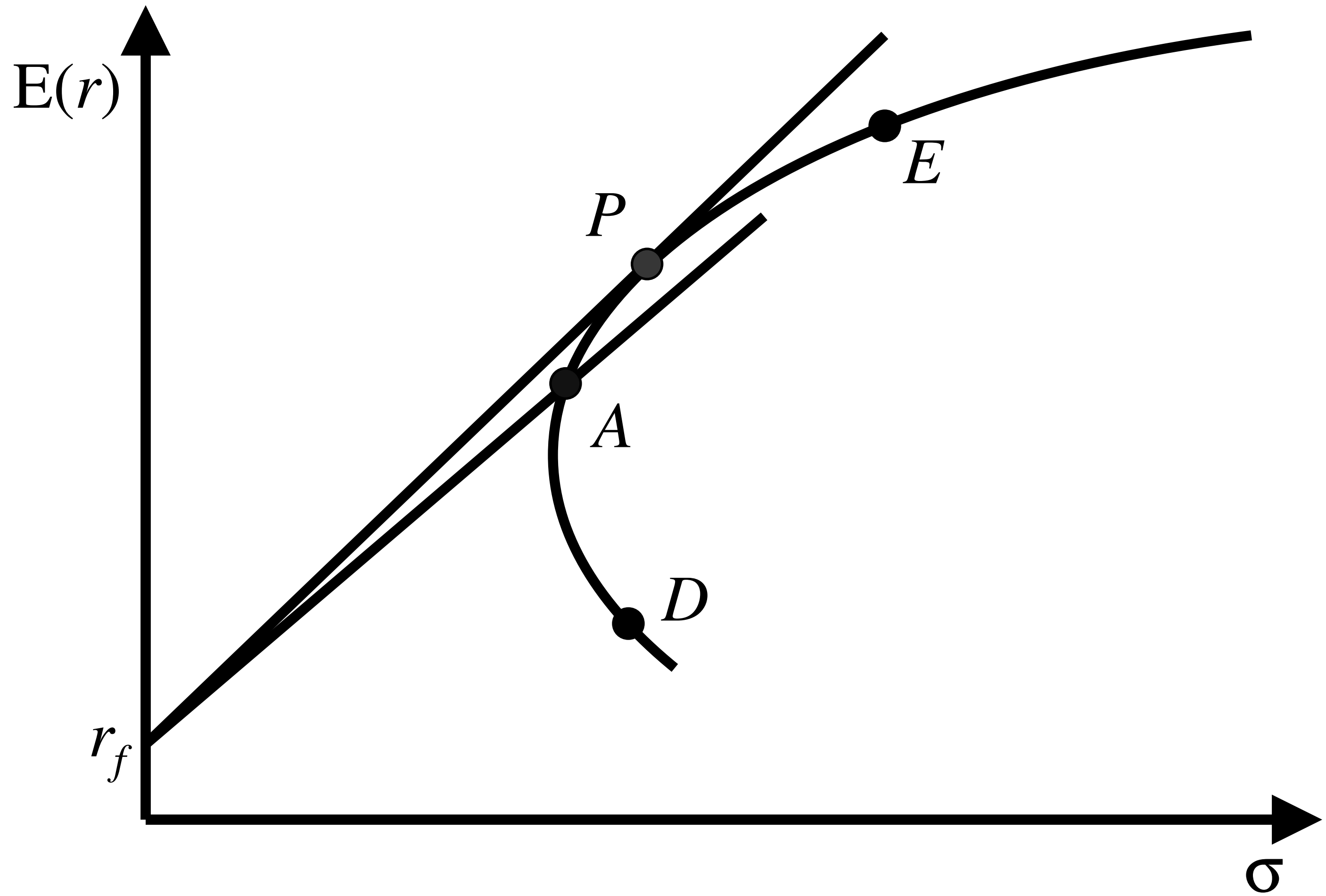


Three-Security Portfolios

- add a risk-free asset
- the CAL depends on the portfolio chosen as *the* risky portfolio
- remember the slope of the CAL (*Sharpe ratio*)

$$S_A = \frac{E(r_A) - r_f}{\sigma_A}$$

Portfolio Opportunity Set with a Risk-free Asset



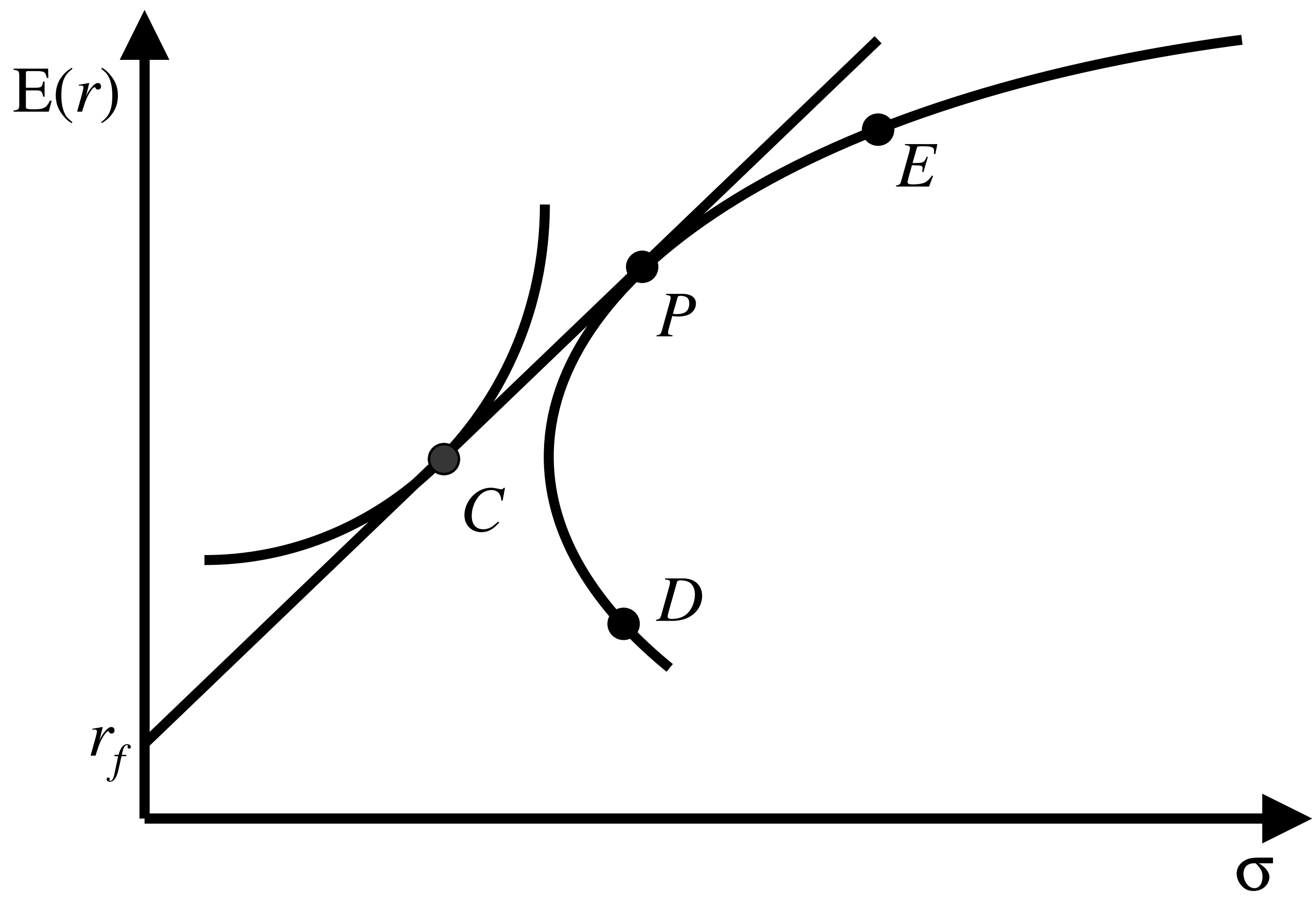
Optimal Risky Portfolio

- optimal choices of *complete* portfolios lie on the highest possible CAL
- hence, optimal risky portfolio is the tangency point of the highest CAL to the portfolio opportunity set
- the highest CAL also has the highest slope
- thus, the investor maximizes the slope:

$$\max_{w_i} S_p = \frac{E(r_p) - r_f}{\sigma_p}$$

- optimal complete portfolio given by investor indifference curve

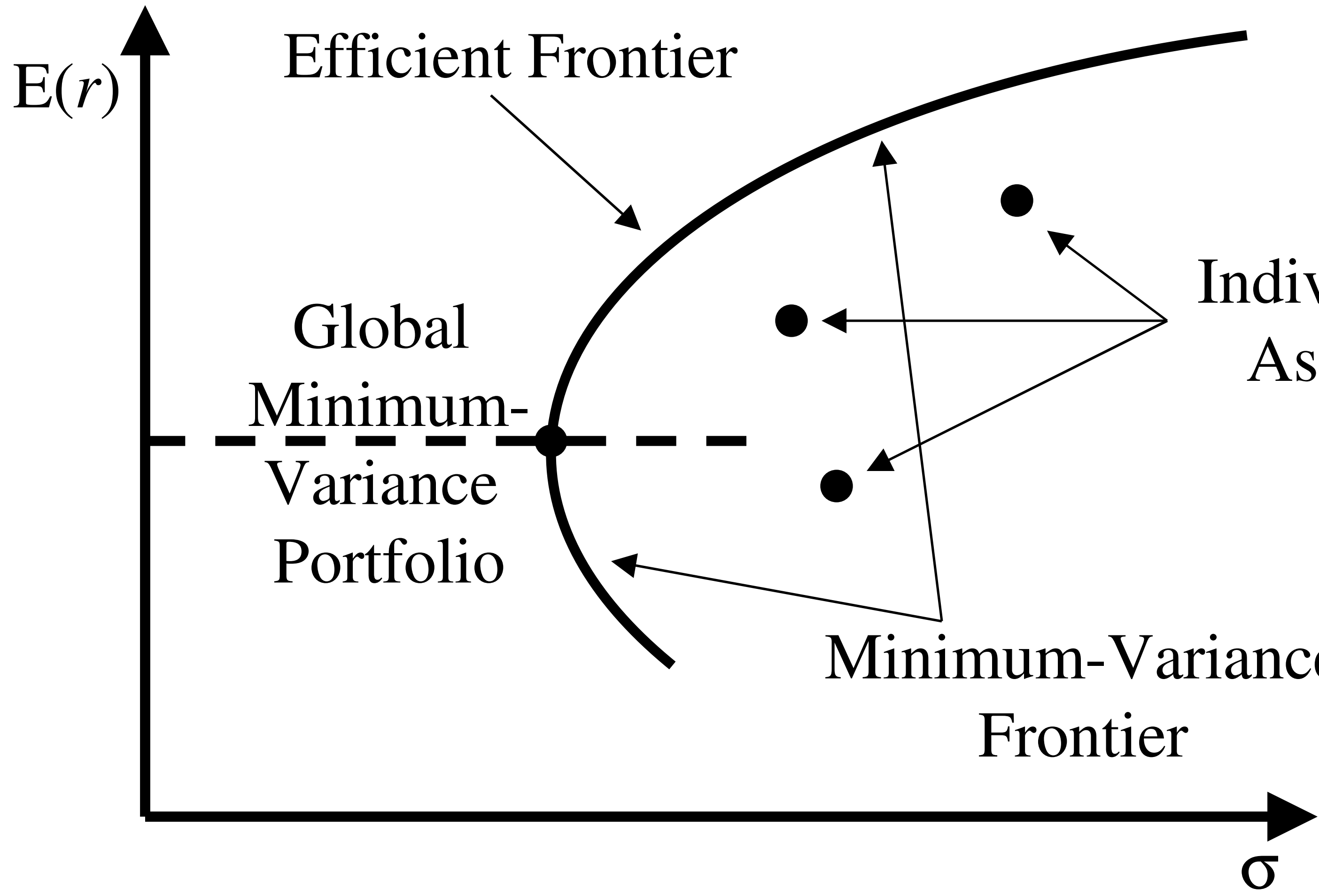
Optimal Complete Portfolio



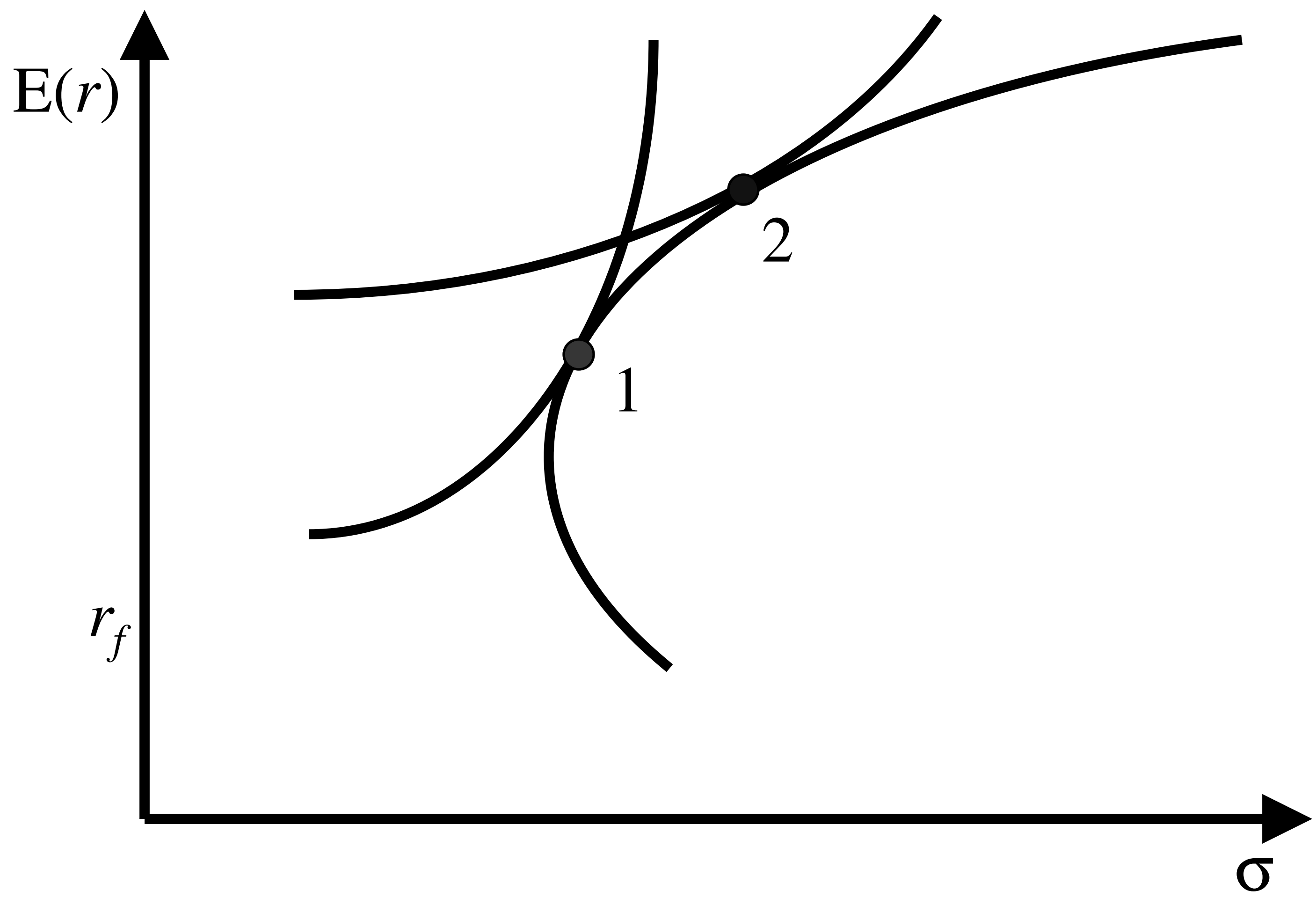
Markovitz Portfolio Selection Model

- many risky assets and a risk-free asset
- *minimum-variance frontier* = the set of portfolios with the lowest variance given an expected rate of return
- *efficient frontier* = the set of portfolios on the minimum-variance frontier, with expected return higher than that of the global minimum variance portfolio
- when short-sales are allowed, all individual assets lie *inside* the minimum-variance frontier

Minimum-Variance Frontier



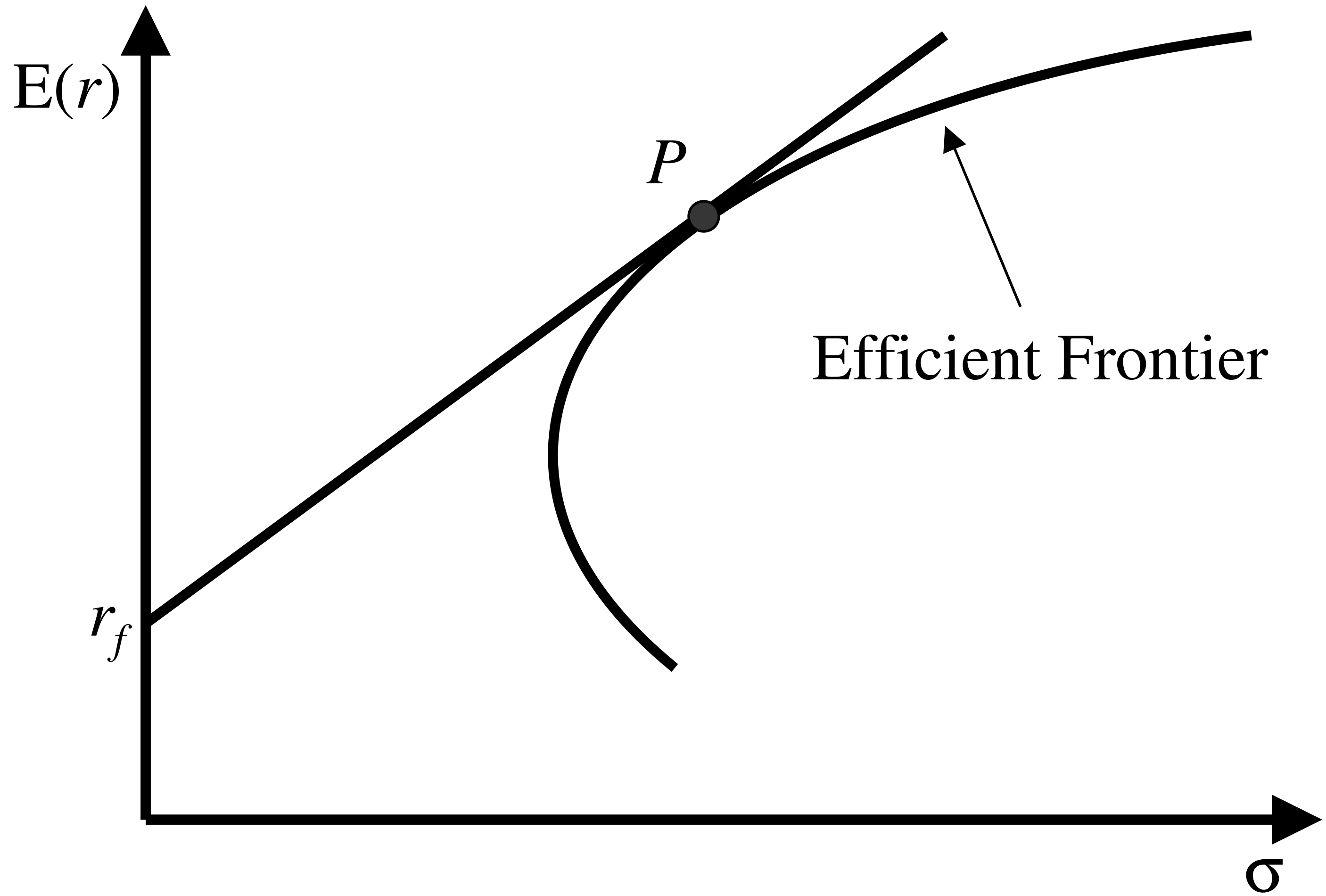
Portfolio Selection without a Risk-free Asset



Capital Allocation with Many Risky Assets

- add a risk-free asset
- again, the CAL depends on the portfolio chosen as *the* risky portfolio
- investors will choose the highest CAL, i.e., the CAL tangent to the efficient frontier
- again, this portfolio is the solution to the optimization problem of maximizing the slope of the CAL

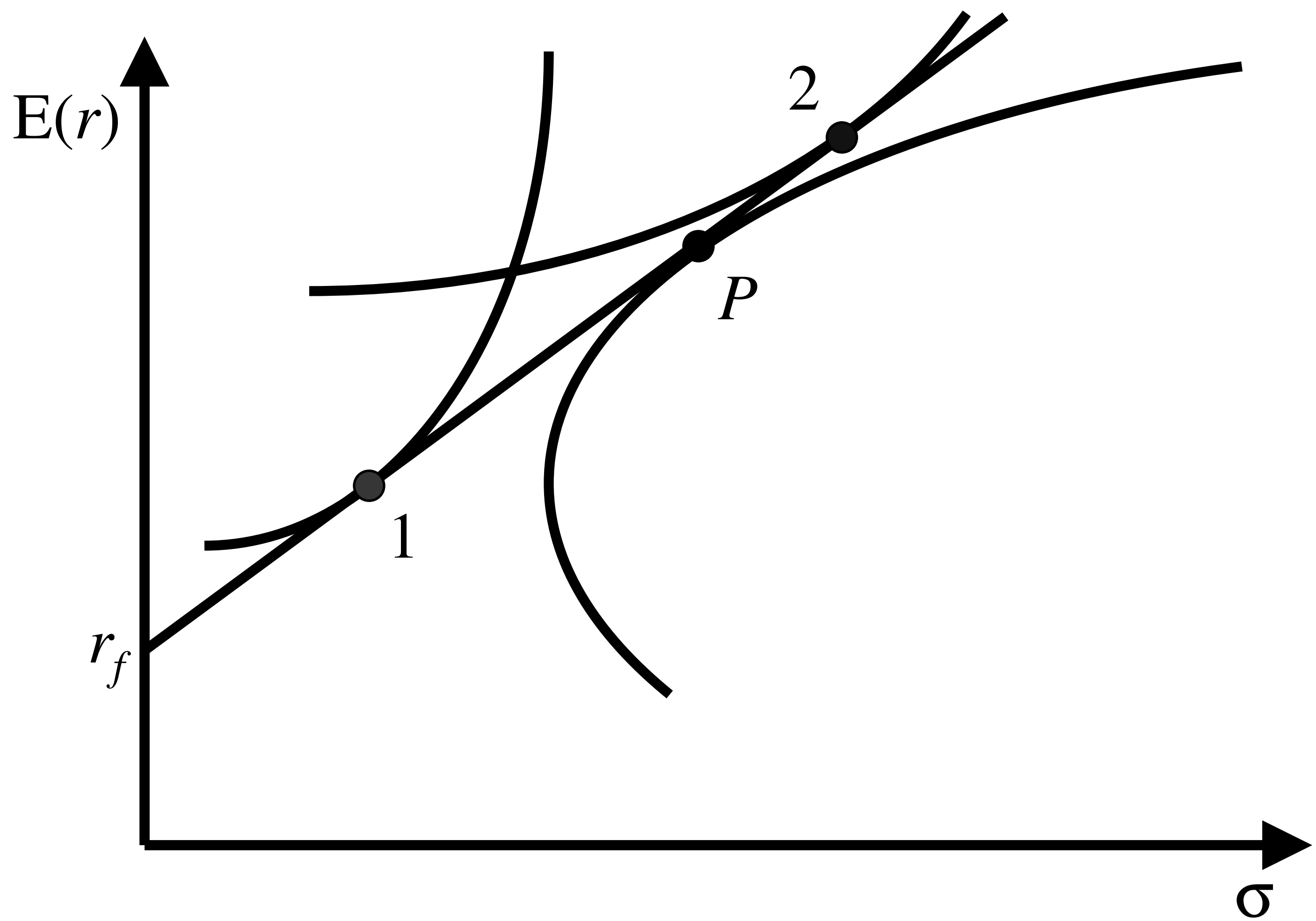
Capital Allocation Line with Many Risky Assets



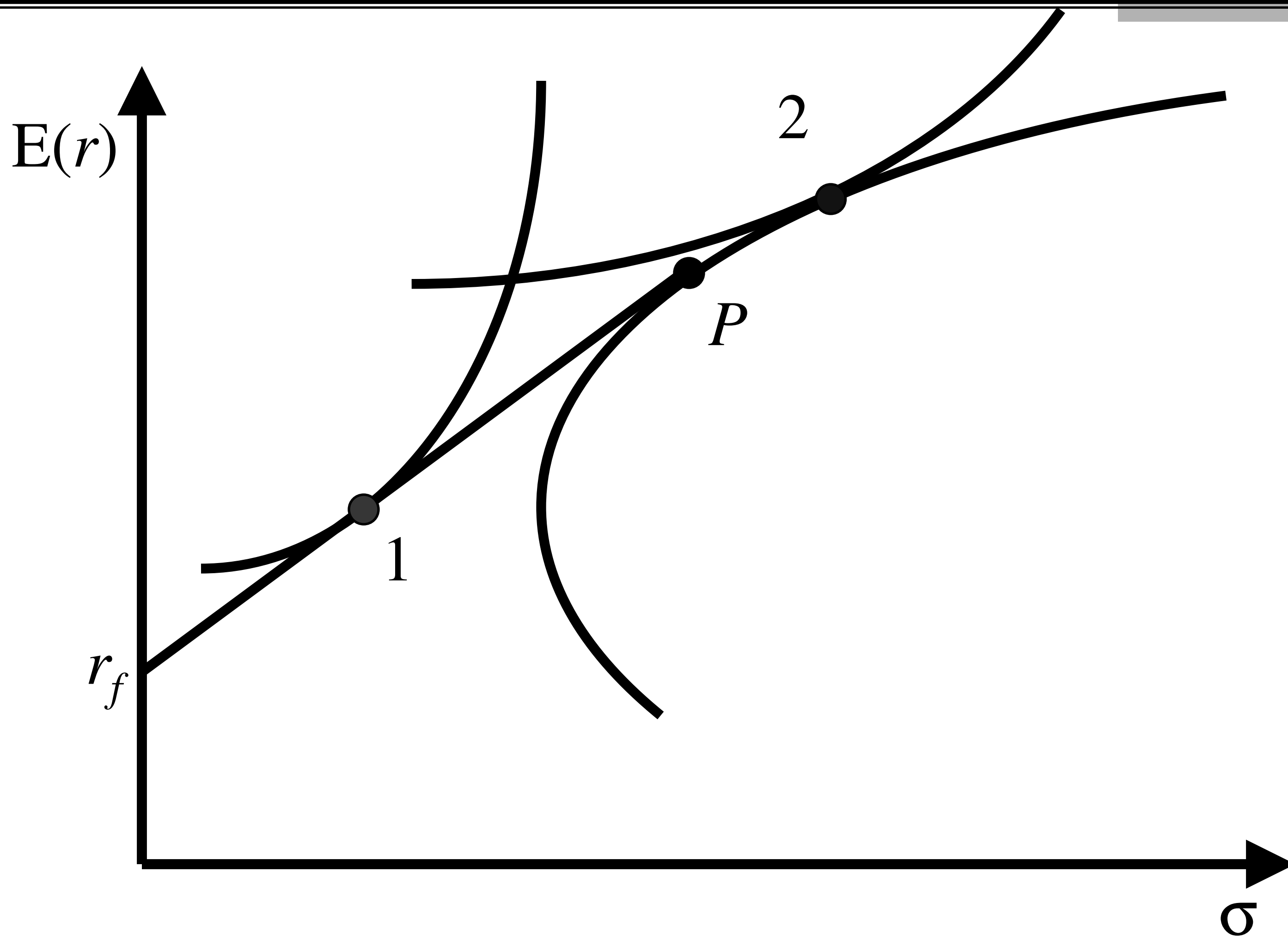
Separation Property

- note that *all* investors will choose the *same* portfolio of risky assets (say P) for their capital allocation problem
- this leads to the *separation property* = the portfolio choice problem can be broken down into two tasks:
  - choosing P, a technical matter (can be done by the broker)
  - deciding on the proportion to be invested in P and in the risky asset (decided by the client)

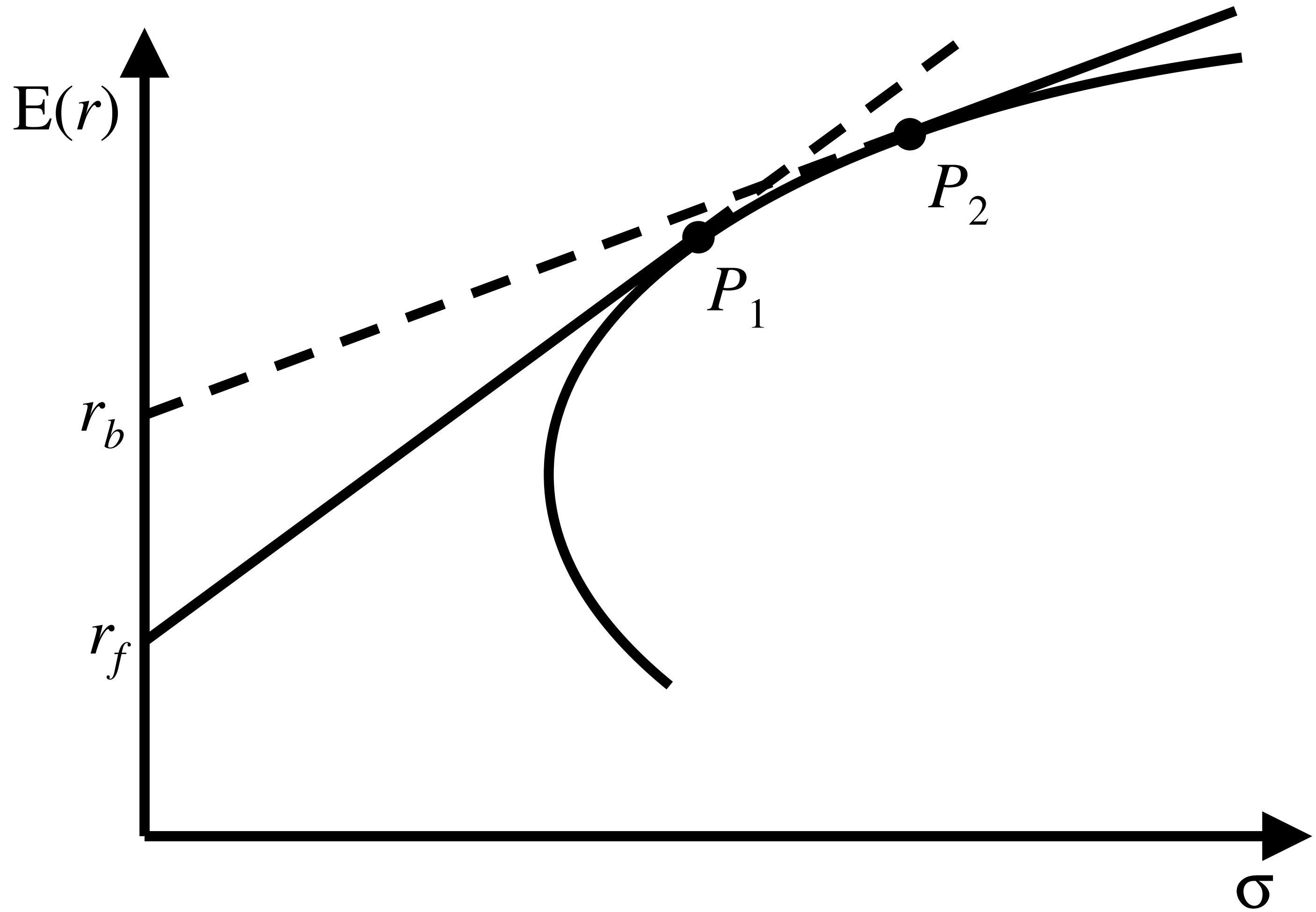
Portfolio Selection with a Risk-free Asset



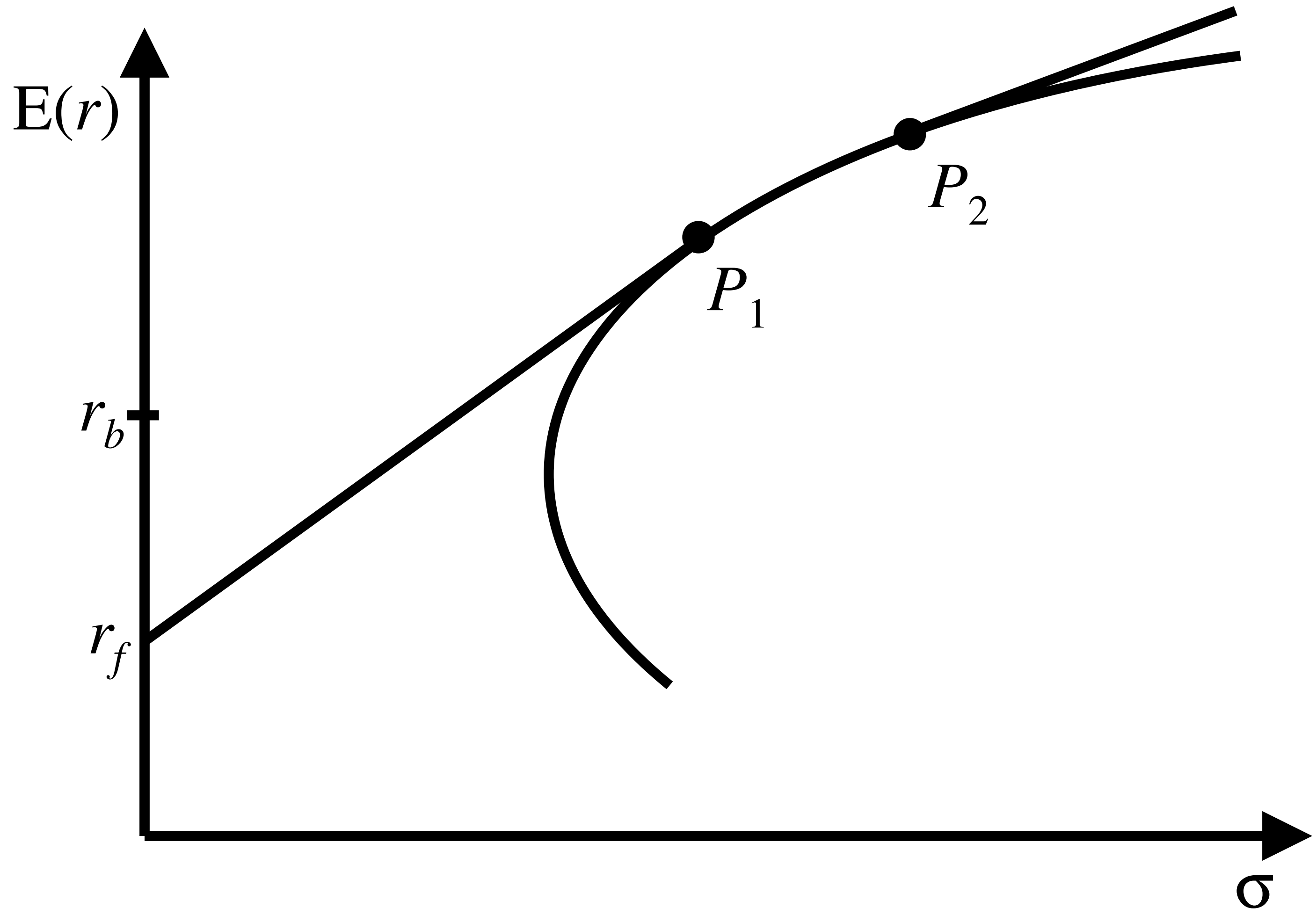
Portfolio Selection with Borrowing Constraints



Investment Opportunity Set with Different Borrowing Rate



Investment Opportunity Set with Different Borrowing Rate (cont.)



Portfolio Selection with Different Borrowing Rate