Index Models

Chapter 10
The Need for a “Simpler” Model

- the input list (i.e., list of assets on the market) in the Markovitz portfolio selection model is very important, determining the accuracy of finding “efficient” portfolios
- however, it involves a lot of calculations
- for example, for $n = 50$ assets we need to calculate (or, more precisely, estimate):
  - $n = 50$ expected returns
  - $n = 50$ variances
  - $n(n - 1) / 2 = 1225$ covariances
- in total, $n(n + 3) / 2 = 1325$ estimates!
The Need for a “Simpler” Model (cont.)

- also, because we estimate returns, variances and covariances, small errors can have large effects
- for example, if we estimate wrongly the covariance matrix (mutually inconsistent correlation coefficients), it is possible that the variance of the portfolio we construct is negative!
- hence, the need for a simpler model, that doesn’t rely on calculations that many calculations
The Single-Factor Model

- Security returns tend to move together because of market risk.
- Suppose we can summarize all the common effects into one macroeconomic variable.
- Then we can write the return on stock $i$ as
  \[
  r_i = E(r_i) + m_i + e_i
  \]
  where $m_i$ is the unanticipated effect of the common macroeconomic factors, and $e_i$ is the unanticipated effect of firm-specific factors.
- Note that $E(m_i) = E(e_i) = 0$. 
The Single-Factor Model (cont.)

- different firms have different sensitivities to macroeconomic events
- denote the sensitivity of firm $i$ to the common set of factors (sensitivity coefficient) by $\beta_i$
- denote the variable that encompasses the unanticipated effect of the common set of macroeconomic factors by $F$
- then $m_i = \beta_i F$ and the equation for the return on stock $k$ becomes the single-factor model:

$$r_i = E(r_i) + \beta_i F + e_i$$
The Single-Index Model

- now, we need a measure for $F$, the common macroeconomic factors
- since a market index corresponds to a well-diversified portfolio, its return should respond only to the common macroeconomic factors
- hence, we can use a market index (say, S&P 500) to approximate our macroeconomic variable → the single-index model
- investors are more interested in risk premiums rather than returns
The Single-Index Model (cont.)

- then we can write the return on stock \( k \) as
  \[
  r_i - r_f = \alpha_i + \beta_i [r_m - r_f] + e_i
  \]
or:
  \[
  R_i = \alpha_i + \beta_i R_m + e_i
  \]
where \( R_i, R_m \) are excess returns

- we can decompose the excess return on a security into three components:
  - \( \alpha_i \) = return if the excess return on the market portfolio is zero
  - \( \beta_i [r_m - r_f] \) = return due to market movements
  - \( e_i \) = return due to unexpected firm-specific factors
Why Beta?

- covariance between returns on stock \( k \) and market portfolio (index):

\[
Cov(r_i, r_m) = Cov(r_i - r_f , r_m) = Cov(r_i - r_f , r_m - r_f )
= Cov(R_i, R_m)
= Cov(\alpha_i + \beta_i R_m + e_i, R_m)
= Cov(\alpha_i, R_m) + Cov(\beta_i R_m, R_m) + Cov(e_i , R_m)
= 0 + \beta_i Cov(R_m, R_m) + 0
= \beta_i \sigma_m^2
\]

- hence, \( \beta_i = \frac{Cov(r_i, r_m)}{\sigma_m^2} \)
Return Variance and Covariances

- note that the variance of returns is

\[ \sigma_i^2 = \text{Var}(\alpha_i + \beta_i R_m + e_i) = \beta_i^2 \sigma_m^2 + \sigma_{ei}^2 \]

- hence, there are two sources of risk:
  - market (idiosyncratic) risk: \( \beta_i^2 \sigma_m^2 \)
  - firm-specific risk: \( \sigma_{ei}^2 \)

- covariance between returns on stocks \( i \) and \( j \):

\[
\text{Cov}(r_i, r_j) = \text{Cov}(R_i, R_j) \\
= \text{Cov}(\alpha_i + \beta_i R_m + e_i, \alpha_j + \beta_j R_m + e_j) \\
= \beta_i \beta_j \sigma_m^2
\]
Why Is It a Simpler Model?

- the input list for $n = 50$ assets consists of:
  - $n = 50$ estimates of expected returns
  - $n = 50$ estimates of sensitivity coefficients ($\beta_i$)
  - 1 (one) estimate of the variance of the market portfolio (index)
  - $n = 50$ estimates of firm-specific risks ($\sigma_{ei}^2$)
- in total, $3n + 1 = 151$ estimates, as compared to 1325!
Advantages and Disadvantages

- **Advantages:**
  - easier to generate the Markovitz frontier
  - allows the specialization of security analysts by industry

- **Disadvantages:**
  - overly simplistic decomposition of risk – macro vs. micro (ignores, for example, industry specific events)
  - resulting portfolios might be inefficient
Estimating the Index Model

- the single index model equation has the form of a regression equation:

  \[ R_{it} = \alpha_i + \beta_i R_{mt} + e_{it} \]

- this defines a line with intercept \( \alpha_i \) and slope \( \beta_i \), with \( e_{it} \) being the deviations from the line for the individual returns

- this line is called the *Security Characteristic Line* (SCL)

- it can be estimated using standard estimation techniques
Estimating the Index Model (cont.)

- to do that, follow the steps:
  - gather historical data on stock prices (usually closing price), market index and risk-free asset (T-bills)
  - construct one-period returns (for a one-month or one-week holding periods) for the stock, the market index and the risk-free asset
  - this yields the variance of the return on the market index
  - construct excess returns for the stock and the market index
  - estimate the index model equation and obtain estimates of $\alpha_i$, $\beta_i$, and $\sigma_{ei}^2$
Security Characteristic Line

\[ R_i, R_m, SCL, \alpha_i \]
Portfolio Risk

- the single index model equation for a portfolio has the same form:

\[ R_p = \alpha_p + \beta_p R_m + e_p \]

where \( \alpha_p, \beta_p, \) and \( e_p \) are weighted returns of the individual stock counterparts

- the variance of the “firm-specific” term \( e_p \) decreases as the number of stocks included in the portfolio increases

- this is another example of the effects of diversification on risk
The Effect of Diversification on Risk

\[ \sigma_p^2 \] vs. number of assets

Diversifiable risk \[ \beta_p^2 \sigma_m^2 \]

Market risk
Problems with the CAPM

- remember that the CAPM holds that
  \[ E(r_i) = r_f + \beta_i [E(r_m) - r_f] \]
- implication of the CAPM:
  - the market portfolio is efficient
  - relationship between risk and expected returns
- in practice, the CAPM is not directly testable, because it makes prediction about ex ante returns, while we only observe ex post returns
Testing the CAPM using the Index Model

- remember that the beta coefficient in the index model is the same as the beta in the CAPM
- we can write the index model equation as
  \[ r_i - r_f = \alpha_i + \beta_i [r_m - r_f] + e_i \]
- take expectations of both sides:
  \[ E(r_i) - r_f = \alpha_i + \beta_i [E(r_m) - r_f] \]
- according to CAPM, a stock’s \( \alpha \) should be equal to zero, on average
- hence, we should find that our estimates of \( \alpha \)'s are centered around zero (Jensen, 1968)
Estimated Alphas
More Practical Insights

- the beta coefficient, variances of return on market index and of firm-specific deviations can be estimated from historical data
- a source of such information is Merrill Lynch’s Security Risk Evaluation book (*beta book*)
- differences from index model:
  - uses returns rather than excess returns
  - ignores dividends
Adjusted Beta

- estimated beta coefficients tend to move toward one over time
- reasons:
  - average beta for all stocks is 1 (market beta)
  - firms become more diversified over time → they eliminate more of firm-specific risk
- Merrill Lynch calculates an adjusted beta to compensate for this tendency:

\[
\beta^a = \frac{2}{3} \beta^e + \frac{1}{3}
\]

where \( \beta^a \) and \( \beta^e \) are adjusted and estimated betas, respectively
suppose an investor identifies an underpriced portfolio $P (\alpha_p > 0)$ and wants to invest in it
still, if the market as a whole declines, she would still end up losing money
to avoid that, she can construct a *tracking portfolio* $T$, with the following structure:
- a proportion $\beta_p$ in the market index
- a proportion $(1 - \beta_p)$ in the risk-free asset
since $T$ is constructed from the market index and the risk-free asset, its alpha coefficient is zero
next, she can buy $P$ and short-sell $T$ at the same time $\Rightarrow$ eliminates the market risk

still, the investment will yield a return (because of the portfolio $P$'s positive alpha)

note: the portfolio is *not* risk-free – it still has the firm-specific risk

this strategy is what many hedge funds do