Arbitrage Pricing Theory and Multifactor Models of Risk and Return

Chapter 11
Single-Factor Model

- remember the single-factor model:
  \[ r_i = E(r_i) + \beta_i F + e_i \]
  where \( F \) and \( e_i \) have zero mean (as they capture surprise changes in the factors)

- this allows the decomposition of risk into market and firm-specific components

- however, this decomposition is overly simplistic:
  - ignores certain factors (e.g., industry-specific)
  - assumes all stocks respond the same to common factors embodied in \( F \)
Multifactor Models

- we should allow for different stocks to have different sensitivities to different types of market-wide shocks (e.g., inflation, business cycles, interest rates etc.)

- *multifactor models* = models that allow for different sensitivities to different factors

- they can provide a better description of security returns
Example – A Two-Factor Model

- suppose we believe the only macroeconomic sources of risk are business cycles (GDP) and interest rates fluctuations (IR)

- rates of return should then respond to unanticipated changes in both factors:

  \[ r_i = E(r_i) + \beta_{i\text{GDP}} \text{GDP} + \beta_{i\text{IR}} \text{IR} + e_i \]

- the beta coefficients are called factor sensitivities, factor loadings or factor betas

- example: compare the sensitivity of returns on an utility company and an airline company
Determining $E(r_i)$

- Our equation is just a description of returns, there is really no theory behind it
- Where does $E(r)$ come from?
- We need a theory of market equilibrium
- CAPM is a theory of market equilibrium, but it only values aggregated market risk

\[ E(r_i) = r_f + \beta_i [E(r_m) - r_f] \]

Or, if we denote market risk premium by $RP_m$:

\[ E(r_i) = r_f + \beta_i RP_m \]
A Multifactor Approach

■ in the CAPM framework, investors are rewarded only for market (non-diversifiable) risk
■ if we acknowledge the existence of multiple sources of risk, the same logic should apply
■ hence, investors should be rewarded for all types of non-diversifiable risk
■ for our previous two-factor model, this implies:
  \[ E(r_i) = r_f + \beta_{i,GDP} RP_{GDP} + \beta_{i,IR} RP_{IR} \]
■ we are left with finding out how to define and calculate the factor risk premiums
Arbitrage Pricing Theory (APT)

- Assumptions:
  - security returns can be described by a (multi-) factor model
  - there are sufficient securities so that firm-specific (idiosyncratic) risk can be diversified away
  - well-functioning security markets do not allow for persistent arbitrage opportunities
Arbitrage

- *arbitrage* = risk-free profits made by investors by exploiting security mispricing, *without* a net investment

- if security prices allow for arbitrage opportunities, the market is not in equilibrium

- hence, there will be pressures on prices to adjust and eliminate these risk-free profits

- *Law of One Price* = assets that are equivalent in all economically relevant aspects should have the same market price
Arbitrage Opportunities

- Investors want to hold infinite positions in an arbitrage opportunity.
- This should create pressures on prices to go up where they are too low and fall where they are too high.
- In equilibrium, the market should satisfy the no-arbitrage condition.
- Note that there is a fundamental difference between risk-return dominance (CAPM) and arbitrage arguments (APT).
**Well-Diversified Portfolios**

- *well-diversified portfolio* = a portfolio such that the firm-specific component of risk is negligible.
- remember that, in the single-factor model,
  \[ \sigma_p^2 = \beta_p^2 \sigma_F^2 + \sigma_{ep}^2 \]
- for a well-diversified portfolio, \( \sigma_{ep}^2 \) is negligible (almost zero)
- since the mean and variance of \( e_p \) are both (almost) zero, any realization of it should be almost zero
- hence, for a well-diversified portfolio,
  \[ r_i = E(r_i) + \beta_i F \]
Security vs. Well-Diversified Portfolio Returns
Betas and Expected Returns

- since firm-specific risk can be diversified away, investors cannot expect to be compensated for that → only systematic risk should impact expected returns
- arbitrage opportunities:
  - well-diversified portfolios with same betas but different expected returns
  - well-diversified portfolios with risk premiums not proportional to their betas
- hence, all well-diversified portfolios should lie on the same line in the expected return-beta space
Different Expected Returns

\[ E(r_A) = 10\% \]

\[ E(r_B) = 8\% \]
Different Expected Return-Beta Relationship
The One-Factor Security Market Line

- suppose there is only one source of risk, that can be embedded into the market portfolio
- the above argument implies that all well-diversified portfolios lie on the same line
- the market portfolio lies on this line, so the line is defined by
  - intercept equal to $r_f$
  - slope equal to the risk premium on the market portfolio
- hence, a CAPM-like equation:

$$E(r_p) = r_f + \beta_p [E(r_m) - r_f]$$
APT vs. CAPM – Portfolios

- as the previous argument was based on arbitrage opportunities, it is called the *Arbitrage Pricing Theory*
- it doesn’t need the strict assumptions of the CAPM
- it is not based on the market portfolio – can be any well-diversified portfolio ➔ more flexibility
- still, it yields a conclusion similar to the CAPM, at least for well-diversified portfolios
APT vs. CAPM – Individual Securities

- CAPM holds that the same relationship is true in the case of individual securities, while the APT holds it true only for well-diversified portfolios.
- Suppose it does not hold for many securities.
- Then it would be possible to construct a well-diversified portfolio for which the expected return-beta relationship fails.
- Hence, this relationship should hold for almost all individual securities.
Factor Portfolios

- suppose again that we have more than one “market risk” factor:

\[ r_i = E(r_i) + \beta_{i1} F_1 + \beta_{i2} F_2 + e_i \]

- factor portfolio = a well-diversified portfolio that has a beta coefficient equal to one for a specific factor and zero for all other factors

- since there are many securities, such portfolios can be constructed for each factor
Multifactor APT

- Let portfolio 1 be a factor portfolio for factor 1, and portfolio 2 be a factor portfolio for factor 2.
- For any well-diversified portfolio $P$, with factor loads $\beta_{p1}$ and $\beta_{p2}$, respectively, we can construct a tracking portfolio using the two factor portfolios:
  - Portfolio 1 has weight $\beta_{p1}$
  - Portfolio 2 has weight $\beta_{p2}$
  - Risk-free asset has weight $(1 - \beta_{p1} - \beta_{p2})$
- Then, the no-arbitrage condition implies that
  \[ E(r_p) = r_f + \beta_{p1} [E(r_1) - r_f] + \beta_{p2} [E(r_2) - r_f] \]
Multifactor Security Market Line

- This equation is just a generalization of the CAPM equation, allowing for more than just one risk factor.
- As before, it holds for all well-diversified portfolios and *almost all* individual securities.
- It can be used, as in the case of CAPM, to find the “fair” return (price) on a portfolio.
What Are the Factors?

- APT does not tell us which factors are relevant
- Previous research suggests:
  - Change in industrial production
  - Change in expected inflation
  - Change in unanticipated inflation
  - Excess return of long-term corporate bonds over long-term government bonds
  - Excess return of long-term government bonds over T-bills
- Possible problem: identifying factors may be hindered by accidental correlations
Intertemporal CAPM (ICAPM)

- the CAPM ignores extra-market hedging needs (e.g., the need of an employee to hedge against labor income risk)
- this could cause the market portfolio not to be the risky optimal portfolio anymore
- Merton showed that these hedging demands lead to a multifactor CAPM model, where an additional risk premium is included in addition to the market portfolio:

\[ E(r_i) = r_f + \beta_{im} [E(r_m) - r_f] + \beta_{ie} [E(r_e) - r_f] \]
ICAPM vs. APT

- ICAPM predicts that sources of risk against which many or dominant investors attempt to hedge will be “priced”
- such sources of risk are: labor income, prices of important consumption goods, changes in future investment opportunities
- as opposed to APT, theory “tells us” what we should look for